Math 534 - Representation Theory III

Quantum groups

New concepts will be written in **bold**, and important new formulas will be $\boxed{\text{boxed}}$.

Material which you have already encountered in Math 314 and 429 will be marked as such.

Details in the proofs that we purposely leave out of the notes, so that you may work out for yourselves, will be colored in blue. Ask your instructors (in person / on the forum) for help.

Table of contents:

Lecture 1

1.1

All our vector spaces will be finite-dimensional and defined over $\mathbb{C}$, unless otherwise stated. Given a vector space V, we have its **tensor algebra**

$$TV = \bigoplus_{k=0}^{\infty} V^{\otimes k} \tag{1}$$

with multiplication given by concatenation of tensors. If we factor the tensor algebra by the two-sided ideal generated by $\{v \otimes v' - v' \otimes v\}_{v,v' \in V}$, we obtain the **symmetric algebra**

$$SV = \bigoplus_{k=0}^{\infty} S^k V \tag{2}$$

while if we factor the tensor algebra by the two-sided ideal generated by $\{v \otimes v' + v' \otimes v\}_{v,v' \in V}$, we obtain the **exterior algebra**

$$\wedge V = \bigoplus_{k=0}^{\infty} \wedge^k V \tag{3}$$

We write elements of $S^k V$ (respectively of $\wedge^k V$) as symmetric (respectively anti-symmetric) tensors

$$v_1 \dots v_k \qquad \text{(respectively } v_1 \wedge \dots \wedge v_k\text{)}$$

for various $v_1, \dots, v_k \in V$. Note that if $\dim V = n$, then $\dim S^k V = \binom{n+k-1}{k}$ and $\dim \wedge^k V = \binom{n}{k}$.

1.2

We will consider complex **Lie algebras** $\mathfrak{g}$, with particular emphasis on [1]

$$\mathfrak{sl}_n = \left\{ X \in \mathrm{Mat}_{n \times n} \,\middle|\, \mathrm{tr}(X) = 0 \right\} \tag{4}$$

We recall from Math 429 that the Lie bracket, i.e. the main structure of a Lie algebra

$$\mathfrak{g} \times \mathfrak{g} \xrightarrow{[\cdot,\cdot]} \mathbb{C}$$

satisfies skew-symmetry and the **Jacobi identity**

$$[x, y] = -[y, x]$$

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

The Lie bracket of $\mathfrak{sl}_n$ is given by the usual commutators of matrices, i.e. $[X, Y] = XY - YX$.

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[1] All matrices in this course will have complex coefficients.

Definition 1. *Given a vector space V, a (Lie algebra) representation $\mathfrak{g} \curvearrowright V$ is an assignment*

$$x \in \mathfrak{g} \quad \rightsquigarrow \quad \phi_x : V \to V$$

which is linear in x and satisfies

$$\phi_{[x,y]} = \phi_x \circ \phi_y - \phi_y \circ \phi_x$$

for all $x, y \in \mathfrak{g}$. In other words, a representation is a Lie algebra homomorphism [2]

$$\mathfrak{g} \to \mathrm{End}(V)$$

where the right-hand side is made into a Lie algebra by commutators of endomorphisms.

Given representations $\mathfrak{g} \curvearrowright V, W$, there are actions

$$\mathfrak{g} \curvearrowright V \oplus W \tag{5}$$
$$\mathfrak{g} \curvearrowright V \otimes W \tag{6}$$
$$\mathfrak{g} \curvearrowright V^* \tag{7}$$

induced by the formulas

$$x \cdot (v, w) = (x \cdot v, x \cdot w) \tag{8}$$
$$x \cdot (v \otimes w) = (x \cdot v) \otimes w + v \otimes (x \cdot w) \tag{9}$$
$$x \cdot \lambda = \lambda \circ (-x) \tag{10}$$

for all $x \in \mathfrak{g}$, $v \in V$, $w \in W$, $\lambda : V \to \mathbb{C}$. By iterating (9), given $\mathfrak{g} \curvearrowright V$ we have actions

$$\mathfrak{g} \curvearrowright V^{\otimes k},\ S^k V,\ \wedge^k V$$

given by the formula

$$x \cdot (v_1 \otimes \cdots \otimes v_k) = \sum_{i=1}^{k} v_1 \otimes \cdots \otimes v_{i-1} \otimes (x \cdot v_i) \otimes v_{i+1} \otimes \cdots \otimes v_k \tag{11}$$

for all $x \in \mathfrak{g}$ and $v_1, \dots, v_k \in V$ (and similarly for symmetric and anti-symmetric tensors).

1.3

For any complex Lie algebra $\mathfrak{g}$, we recall its **universal enveloping algebra**

$$\boxed{U\mathfrak{g} = T\mathfrak{g} \Big/ \Big(x \otimes y - y \otimes x - [x, y] \Big)_{x,y \in \mathfrak{g}}} \tag{12}$$

In other words, an element of $U\mathfrak{g}$ is an arbitrary linear combination of tensors of elements of $\mathfrak{g}$, modulo relations of the form

$$t_1 \otimes x \otimes y \otimes t_2 - t_1 \otimes y \otimes x \otimes t_2 - t_1 \otimes [x, y] \otimes t_2 \tag{13}$$

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[2] Recall that a Lie algebra homomorphism $\mathfrak{g} \to \mathfrak{g}'$ is a linear map which respects the Lie brackets.

for any tensors $t_1, t_2 \in T\mathfrak{g}$ and any $x, y \in \mathfrak{g}$. The effect this has on $U\mathfrak{g}$ is to ensure that the Lie bracket $[x, y]$ is identified with the commutator $x \otimes y - y \otimes x$ in all formulas. While the construction of the universal enveloping algebra might seem a little dry, it is motivated by the formal fact that

$$\mathfrak{g} \curvearrowright V \text{ is a Lie algebra representation} \qquad \Leftrightarrow \qquad U\mathfrak{g} \curvearrowright V \text{ is an algebra module}$$

(all our algebras will be unital, and "module" henceforth refers to left modules). Because relations (13) are not homogeneous in the number of tensor factors, the universal enveloping algebra does not inherit the grading of $T\mathfrak{g}$ by number of tensor factors. However, it is a **filtered** algebra

$$0 = U_{-1}\mathfrak{g} \subset U_0\mathfrak{g} \subset U_1\mathfrak{g} \subset \cdots \subset U_k\mathfrak{g} \subset \cdots \subset U\mathfrak{g} \quad \text{such that } U\mathfrak{g} = \bigcup_{k=0}^{\infty} U_k\mathfrak{g} \tag{14}$$

(where $U_k\mathfrak{g}$ is the linear span of all tensors of length $\leq k$) meaning that we have $U_k\mathfrak{g} \cdot U_\ell\mathfrak{g} \subseteq U_{k+\ell}\mathfrak{g}$ for all $k, \ell \geq 0$. Thus, we may construct the **associated graded** algebra

$$\mathrm{gr}(U\mathfrak{g}) = \bigoplus_{k=0}^{\infty} U_k\mathfrak{g} \Big/ U_{k-1}\mathfrak{g}$$

which satisfies the following important result called the Poincaré-Birkhoff-Witt (PBW) theorem.

Theorem 1. *There is an isomorphism of graded algebras*

$$S\mathfrak{g} \cong \mathrm{gr}(U\mathfrak{g}) \tag{15}$$

induced by the identity on $\mathfrak{g}$.

As a consequence, the PBW theorem implies that if we have a basis

$$x_1, \ldots, x_d \tag{16}$$

of $\mathfrak{g}$ as a vector space, then the symbols

$$\Big\{ \underbrace{x_1 \otimes \cdots \otimes x_1}_{a_1 \text{ factors}} \otimes \underbrace{x_2 \otimes \cdots \otimes x_2}_{a_2 \text{ factors}} \otimes \cdots \otimes \underbrace{x_d \otimes \cdots \otimes x_d}_{a_d \text{ factors}} \Big\}_{a_1, \ldots, a_d \geq 0} \tag{17}$$

give rise to a basis of $U\mathfrak{g}$.

1.4

We have so far discussed $U\mathfrak{g}$ as an algebra; the fact that the direct sum of algebra modules is an algebra module explains construction (5). But what about tensor products (6) and duals (7), which are constructions that do not make sense for an arbitrary algebra? There must be something about the algebra $U\mathfrak{g}$ that allows us to have the operations

$$\Big(U\mathfrak{g}\text{-module}\Big) \times \Big(U\mathfrak{g}\text{-module}\Big) \xrightarrow{\otimes} \Big(U\mathfrak{g}\text{-module}\Big)$$

and

$$\Big(U\mathfrak{g}\text{-module}\Big) \xrightarrow{*} \Big(U\mathfrak{g}\text{-module}\Big)$$

with all the suitable compatibilities. In fact, this is precisely the structure of a Hopf algebra.

Definition 2. *A **coalgebra** A is a vector space endowed with a counit*

$$\varepsilon : A \to \mathbb{C}$$

and a coproduct

$$\Delta : A \to A \otimes A$$

which satisfy the compatibility conditions

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\varepsilon\otimes\mathrm{Id}_A"] \\
A \otimes A \arrow[r, "\mathrm{Id}_A\otimes\varepsilon"] & A
\end{tikzcd}$$

and the coassociativity

$$\begin{tikzcd}
A \arrow[r, "\Delta"] \arrow[d, "\Delta"'] & A \otimes A \arrow[d, "\Delta\otimes\mathrm{Id}_A"] \\
A \otimes A \arrow[r, "\mathrm{Id}_A\otimes\Delta"] & A \otimes A \otimes A
\end{tikzcd}$$

Note that the construction above is simply the dual notion to that of an algebra; just reverse all the arrows and you obtain the usual structures and axioms of an algebra. Therefore

$$\boxed{\text{if } A \text{ is an algebra, then } A^* = \{\lambda : A \to \mathbb{C}\} \text{ is a coalgebra}} \tag{18}$$

Recall that if A is an algebra, then the opposite algebra is

$$A^{\mathrm{op}} = A \text{ endowed with the opposite product}$$

Similarly, the opposite of a coalgebra A is

$$A^{\mathrm{coop}} = A \text{ endowed with the opposite coproduct}$$

Here, the **opposite** coproduct refers to

$$\Delta^{\mathrm{op}} = \mathrm{swap} \circ \Delta \tag{19}$$

with "swap" referring to the permutation of factors of $A \otimes A$. Just like an algebra is called commutative when $A = A^{\mathrm{op}}$, a coalgebra is called **cocommutative** when $A = A^{\mathrm{coop}}$.

Remark. *It is often convenient to use **Sweedler notation** for the coproduct*

$$\Delta(a) = a_1 \otimes a_2 \tag{20}$$

instead of the more appropriate (but cumbersome) notation $\Delta(a) = a_1 \otimes a_2 + a_1' \otimes a_2' + a_1'' \otimes a_2'' + \ldots$. For example, with this notation the opposite coproduct takes the form $\Delta^{\mathrm{op}}(a) = a_2 \otimes a_1$.

1.5

Definition 3. *A* ***bialgebra*** *is a vector space which is both an algebra and a coalgebra in a compatible way, which means that the counit and coproduct are algebra homomorphisms, i.e.*

$$\varepsilon(ab) = \varepsilon(a)\varepsilon(b) \tag{21}$$

$$\Delta(ab) = \Delta(a)\Delta(b) \tag{22}$$

for all $a, b \in A$, where the multiplication of tensors is done component-wise. Thus, condition (22) *can be expressed in Sweedler notation* (20) *as $(ab)_1 = a_1 b_1$ and $(ab)_2 = a_2 b_2$.*

If A is a bialgebra, one can construct three other bialgebras

$$A^{\text{op}}, \; A^{\text{coop}}, \; A^{\text{op,coop}}$$

by replacing the product by its opposite, the coproduct by its opposite, or both.

The analogue of (18) is

$$\boxed{\text{if } A \text{ is a bialgebra, then } A^* \text{ is also a bialgebra}} \tag{23}$$

with the product (respectively coproduct) of A^* dual to the coproduct (respectively product) of A.

Definition 4. *A* ***Hopf algebra*** *is a bialgebra A endowed with an invertible*[3] *map called* ***antipode***

$$S : A \to A \tag{24}$$

which must be an algebra anti-automorphism

$$S(1) = 1, \quad S(ab) = S(b)S(a), \quad \forall a, b \in A \tag{25}$$

a coalgebra anti-automorphism

$$\varepsilon \circ S = S, \quad S(a_1 \otimes a_2) = S(a_2) \otimes S(a_1), \quad \forall a \in A \tag{26}$$

and furthermore satisfies

$$S(a_1)a_2 = a_1 S(a_2) = \varepsilon(a)1, \quad \forall a \in A \tag{27}$$

Please show that if A is a Hopf algebra, then so are A^{op}, A^{coop}, $A^{\text{op,coop}}$. Linear maps $A \to B$ are called algebra/coalgebra/bialgebra/Hopf algebra **homomorphisms** if they preserve the respective structures. The following is a straightforward, but important exercise, which we leave to you.

Proposition 1. *For any Lie algebra $\mathfrak{g}$, its universal enveloping algebra is a Hopf algebra via*

$$\Delta(x) = x \otimes 1 + 1 \otimes x$$

$$S(x) = -x, \quad \varepsilon(x) = 0$$

for all $x \in \mathfrak{g}$.

Strictly speaking, statements (18) and (23) require A to be finite-dimensional. However, the all-too-important setting of Proposition 1 involves an infinite-dimensional algebra. With this in mind, in the Exercise Sheet you will see how to make A^* into a Hopf algebra even when A is an infinite-dimensional Hopf algebra, under certain reasonable assumptions.

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[3] Some authors do not require the antipode to be invertible, though we will for simplicity. Larson-Sweedler showed that the antipode is automatically invertible if A is finite-dimensional, and Skryabin conjectured that it is automatically invertible if A is a noetherian ring.

1.6

With Proposition 1 in mind, then the constructions (6) and (7) become particular cases of the following general construction.

Definition 5. *For any Hopf algebra A and any modules $A \curvearrowright V, W$, there are actions*

$$A \curvearrowright V \oplus W \tag{28}$$

$$A \curvearrowright V \otimes W \tag{29}$$

$$A \curvearrowright V^* \tag{30}$$

induced by the formulas

$$a \cdot (v, w) = (a \cdot v, a \cdot w) \tag{31}$$

$$a \cdot (v \otimes w) = (a_1 \cdot v) \otimes (a_2 \cdot w) \tag{32}$$

$$a \cdot \lambda = \lambda \circ S(a) \tag{33}$$

for all $a \in A$, $v \in V$, $w \in W$, $\lambda : V \to \mathbb{C}$. We also have the trivial module

$$A \curvearrowright \mathbb{C}, \quad a \text{ acts by multiplication with } \varepsilon(a), \quad \forall a \in A$$

If A is only a bialgebra and not a Hopf algebra, then we only have the operations (28) and (29).

Theorem 2. *If A is a Hopf algebra, the natural maps yield A-module isomorphisms*

$$U \otimes (V \otimes W) \cong (U \otimes V) \otimes W \tag{34}$$

$$V \otimes \mathbb{C} \cong V \cong \mathbb{C} \otimes V \tag{35}$$

$$U \otimes (V \oplus W) \cong (U \otimes V) \oplus (U \otimes W) \tag{36}$$

$$(V \oplus W) \otimes U \cong (V \otimes U) \oplus (W \otimes U) \tag{37}$$

$$(V \otimes W)^* \cong W^* \otimes V^* \tag{38}$$

$$\mathbb{C}^* \cong \mathbb{C} \tag{39}$$

Moreover, the natural unit and trace maps

$$\mathbb{C} \longrightarrow V \otimes V^*, \qquad V^* \otimes V \longrightarrow \mathbb{C} \tag{40}$$

are also A-module homomorphisms.

Proof. Formulas (34) and (35) are consequences of the coassociativity and counit properties of the coproduct. Formulas (36) and (37) are consequences of the way direct sums and tensor products work. Formulas (38) and (39) are consequences of the fact that the antipode is a coalgebra anti-automorphism. Finally, (40) follows from (27).

□

Remark. *Note that the usual permutation of factors does <u>not</u> give an A-module isomorphism*

$$V \otimes W \ncong W \otimes V \tag{41}$$

for general A-modules V, W, unless A is a cocommutative Hopf algebra. This seeming failure will be the motivation for developing the theory of R-matrices, one of our main goals in this course, which provide non-trivial isomorphisms (41) of modules. Moreover, the identity map does not provide an isomorphism of A-modules

$$V^{**} \cong V$$

even for finite-dimensional V, unless we have $S^2 = \mathrm{Id}_A$.